

n -RELATIVE CATEGORIES: A MODEL FOR THE HOMOTOPY THEORY OF n -FOLD HOMOTOPY THEORIES

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ABSTRACT. We introduce, for every integer $n \geq 1$, the notion of an n -relative category and show that the category of the small n -relative categories is a model for the homotopy theory of n -fold homotopy theories, i.e. homotopy theories of ... of homotopy theories.

1. BACKGROUND AND MOTIVATION

In this introduction we

- recall some results of (higher) homotopy theory, and
- explain how they led to the current manuscript.

We start with

1.1. Rezk and re-Rezk. In [R] Charles Rezk constructed a left Bousfield localization of the Reedy structure on the category \mathbf{sS} of small simplicial spaces (i.e. bisimplicial sets) and showed it to be a model for the *homotopy theory of homotopy theories*.

Furthermore it was noted in [B] (and a proof thereof can be found in [Lu, §1]) that iteration of Rezk's construction yields, for every integer $n > 1$, a left Bousfield localization of the Reedy structure on the category $\mathbf{s}^n\mathbf{S}$ of small n -simplicial spaces (i.e. $(n+1)$ -simplicial sets) which is a model for the *homotopy theory of n -fold homotopy theories*, i.e. homotopy theories of ... of homotopy theories.

We will call the weak equivalences in these left Bousfield localization (which are often referred to as complete Segal equivalences) just *Rezk equivalences*.

Rezk's original result also gave rise to the following result on

1.2. Relative categories. Recall that a *relative category* is a pair $(\mathbf{C}, w\mathbf{C})$ consisting of a category \mathbf{C} and a subcategory $w\mathbf{C} \subset \mathbf{C}$ which contains all the objects of \mathbf{C} and of which the maps are called *weak equivalences*.

Then it was shown in [BK] that Rezk's model structure on \mathbf{sS} (1.1) can be lifted to a Quillen equivalent *Rezk structure* on the category \mathbf{RelCat} of the small relative categories, the weak equivalences of which will also (1.1) be called *Rezk equivalences*.

The category \mathbf{RelCat} is connected to \mathbf{sS} by a *simplicial nerve functor* $N: \mathbf{RelCat} \rightarrow \mathbf{sS}$ with the property that a map $f \in \mathbf{RelCat}$ is a Rezk equivalence iff the map $Nf \in \mathbf{sS}$ is so. Moreover if we denote by \mathbf{Rk} the subcategories of the Rezk equivalences in both \mathbf{RelCat} and \mathbf{sS} , then the simplicial nerve functor has the property that

(i) *the relative functor*

$$N: (\mathbf{RelCat}, \mathbf{Rk}) \longrightarrow (\mathbf{sS}, \mathbf{Rk})$$

is a *homotopy equivalence of relative categories*, in the sense that there exists a relative functor

$$M: (\mathbf{sS}, \mathbf{Rk}) \longrightarrow (\mathbf{RelCat}, \mathbf{Rk})$$

called a *homotopy inverse* of N such that the compositions MN and NM can be connected to the identity functors of \mathbf{RelCat} and \mathbf{sS} by finite zigzags of natural weak equivalences.

This in turn implies that

(ii) *the relative category $(\mathbf{RelCat}, \mathbf{Rk})$ is, just like $(\mathbf{sS}, \mathbf{Rk})$, a model for the homotopy theory of homotopy theories.*

The proof of all this is essentially a relative version of the proof of the following classical result of Bob Thomason.

1.3. Thomason's result. In [T] Bob Thomason lifted the usual model structure on the category \mathbf{S} of small spaces (i.e. simplicial sets) to a Quillen equivalent one on the category \mathbf{Cat} of small categories and noted that these two categories were connected by the *nerve functor* $N: \mathbf{Cat} \rightarrow \mathbf{S}$ which has the property that a map $f \in \mathbf{Cat}$ is a weak equivalence iff $Nf \in \mathbf{S}$ is so. It follows that, if \mathbf{W} denotes the categories of weak equivalences in both \mathbf{Cat} and \mathbf{S} , then

(i) *the relative functor $N: (\mathbf{Cat}, \mathbf{W}) \rightarrow (\mathbf{S}, \mathbf{W})$ is a homotopy equivalence of relative categories (1.2(i))*

which in turn implies that

(ii) *the relative category $(\mathbf{Cat}, \mathbf{W})$ is, just like (\mathbf{S}, \mathbf{W}) a model for the theory of homotopy types.*

His proof was however far from simple as it involved notions like two-fold subdivision and so-called Dwyer maps.

We end with recalling

1.4. A result of Dana Latch. In [La] Dana Latch noted that, if one just wanted to prove 1.3(i) and 1.3(ii), one could do this by an argument that was much simpler than Thomason's and that, instead of the cumbersome two-fold subdivisions and Dwyer maps, involved the rather natural notion of the *category of simplices of a simplicial set*.

Now we can finally discuss

1.5. The current paper. The results mentioned in 1.1 and 1.2 above suggest that, for every integer $n > 1$, there might exist some generalization of the notion of a relative category such that the category of such generalized relative categories admits a model structure which is Quillen equivalent to the Rezk structure on $\mathbf{s}^n \mathbf{S}$.

As however we did not see how to attack this question we turned to a much simpler one suggested by the result of Dana Latch that was mentioned in 1.4 above, namely to prove 1.2(i) directly by showing that

- the *simplicial nerve functor*

$$N: (\mathbf{RelCat}, \mathbf{Rk}) \longrightarrow (\mathbf{sS}, \mathbf{Rk})$$

has an appropriately defined *relative category of bisimplices functor*

$$\Delta_{\text{rel}}: (\mathbf{sS}, \mathbf{Rk}) \longrightarrow (\mathbf{RelCat}, \mathbf{Rk})$$

as a homotopy inverse.

It turned out that not only could we do this, but the relative simplicity of our proof suggested that a similar proof might work for appropriately generalized relative categories. And indeed, after the necessary trial and error and frustration, we discovered a notion of what we will call *n-relative categories* which fitted the bill.

Hence the current manuscript.

2. AN OVERVIEW

2.1. **Summary.** There are five more sections.

- In the first (§3) we introduce *n-relative categories*.
- In the second (§4) we investigate an *adjunction*

$$K: \mathbf{s}^n \mathbf{S} \longleftrightarrow \mathbf{Rel}^n \mathbf{Cat} : N$$

between the category $\mathbf{s}^n \mathbf{S}$ of small n -simplicial spaces and the category $\mathbf{Rel}^n \mathbf{Cat}$ of small n -relative categories, in which the right adjoint N is the *n-simplicial nerve functor*.

- Next (in §5 and 6) we formulate and prove our main result.
- In an appendix (§7) we mention two relations between the categories $\mathbf{Rel}^n \mathbf{Cat}$ and $\mathbf{Rel}^{n+1} \mathbf{Cat}$.

In more detail:

2.2. **n -Relative categories.** Motivated by the fact that in an n -simplicial space (i.e. an $(n+1)$ -simplicial set), just like in a simplicial space, the “space direction” plays a different role than “the n simplicial directions”, we define (in §3) an *n-relative category* \mathbf{C} as an $(n+2)$ tuple

$$\mathbf{C} = (a\mathbf{C}, v_1\mathbf{C}, \dots, v_n\mathbf{C}, w\mathbf{C})$$

consisting of a category $a\mathbf{C}$ and subcategories $v_1\mathbf{C}, \dots, v_n\mathbf{C}$ and $w\mathbf{C} \subset a\mathbf{C}$ subject to the following conditions:

- Each of the subcategories contains all the objects of $a\mathbf{C}$ and together with $a\mathbf{C}$ they form a commutative diagram with $2n$ arrows of the form

$$\begin{array}{ccc} & w\mathbf{C} & \\ \swarrow & & \searrow \\ v_1\mathbf{C} & \cdots & v_n\mathbf{C} \\ \searrow & & \swarrow \\ & a\mathbf{C} & \end{array}$$

which means that $v_1\mathbf{C}, \dots, v_n\mathbf{C}$ can be considered as n *relative categories* which all have $w\mathbf{C}$ as their *category of weak equivalences*.

- (ii) The purpose of $a\mathbf{C}$, the *ambient category*, is to encode “the extent to which any two of the $v_i\mathbf{C}$ ’s commute” and we therefore impose on $a\mathbf{C}$ two conditions which simultaneously ensure that $a\mathbf{C}$ does not contain any superfluous information and that the associated (see 2.3) n -simplicial nerve functor, just like the classical nerve functor, has a *left adjoint* which is also a *left inverse*.

2.3. The n -simplicial nerve functor. In §4 we introduce an adjunction

$$K: s^n\mathbf{S} \longleftrightarrow \mathbf{Rel}^n\mathbf{Cat} : N$$

between the category $s^n\mathbf{S}$ of the small n -simplicial spaces (i.e. $(n+1)$ -simplicial sets) and the category $\mathbf{Rel}^n\mathbf{Cat}$ of the small n -relative categories (2.2).

If, for every integer $p \geq 0$, \mathbf{p} denotes the category

$$0 \longrightarrow \cdots \longrightarrow p$$

then the left adjoint K is the colimit preserving functor which sends each standard multisimplex $\Delta[p_n, \dots, p_1, q]$ to an n -relative version of the category

$$\mathbf{p}_n \times \cdots \times \mathbf{p}_1 \times \mathbf{q} .$$

The right adjoint N will be referred to as the *n -simplicial nerve functor*.

We also note that the counit and the unit of this adjunction have some nice properties and in particular that *the counit is an isomorphism* (which is equivalent to the statement that “ K is not only a left adjoint of N , but also a left inverse” (cf. 2.2(ii))).

2.4. The main result. To formulate our main result (in §5) we use the n -simplicial nerve functor N (2.3) to lift the Reedy and the Rezk equivalences in $s^n\mathbf{S}$ (1.1) to what we will also call Reedy and Rezk equivalences in $\mathbf{Rel}^n\mathbf{Cat}$ and denote by \mathbf{Ry} and \mathbf{Rk} the subcategories of these Reedy and Rezk equivalences in both $s^n\mathbf{S}$ and $\mathbf{Rel}^n\mathbf{Cat}$.

Our main result then is

Theorem A. *The relative functor*

$$N: (\mathbf{Rel}^n\mathbf{Cat}, \mathbf{Rk}) \longrightarrow (s^n\mathbf{S}, \mathbf{Rk})$$

is a homotopy equivalence of relative categories (1.2(i)).

In view of the fact that the Rezk equivalences in $s^n\mathbf{S}$ are the weak equivalences in a left Bousfield localization of the Reedy structure this theorem is a ready consequence of

Theorem B. *The relative functor*

$$N: (\mathbf{Rel}^n\mathbf{Cat}, \mathbf{Ry}) \longrightarrow (s^n\mathbf{S}, \mathbf{Ry})$$

is a homotopy equivalence of relative categories.

2.5. The proof. Most of the proof of our main result (2.4) is also in §5, except for the proof of two of the propositions involved which we will deal with in §6.

Apart from some properties of the counit and the unit of the adjunction (2.3)

$$K: s^n \mathcal{S} \longleftrightarrow \mathbf{Rel}^n \mathbf{Cat} : N ,$$

the proof involves the rather obvious *category of multisimplices functor*

$$\Delta: s^n \mathcal{S} \longrightarrow \mathbf{Cat} \quad (1.3)$$

and an n -relative version thereof, the *n -relative category of multisimplices functor*

$$\Delta_{\text{rel}}: s^n \mathcal{S} \longrightarrow \mathbf{Rel}^n \mathbf{Cat} ,$$

which will be the required homotopy inverse (1.2(i)) of N .

In particular we need two rather simple properties of Δ , as well as two properties of Δ_{rel} . The proofs of the latter take rather more effort and will therefore be dealt with separately in §6.

2.6. An appendix. In an appendix (§7) we mention two relations between the categories $\mathbf{Rel}^n \mathbf{Cat}$ and $\mathbf{Rel}^{n+1} \mathbf{Cat}$ which one would expect higher homotopy theories to have:

- (i) *That the functor $\mathbf{Rel}^n \mathbf{Cat} \rightarrow \mathbf{Rel}^{n+1} \mathbf{Cat}$ which sends*

$$(aC, v_1 C, \dots, v_n C, wC) \text{ to } (aC, v_1 C, \dots, v_n C, wC, wC)$$

has a right adjoint which is a left inverse.

- (ii) *That every object of $\mathbf{Rel}^{n+1} \mathbf{Cat}$ gives rise to a category enriched over $\mathbf{Rel}^n \mathbf{Cat}$.*

3. n -RELATIVE CATEGORIES

After a brief review of *relative categories* we

- introduce n -relative categories ($n \geq 1$) and
- describe some simple but useful examples which we will need in the next section.

3.1. Relative categories. A **relative category** is a pair (C, W) (often denoted by just C) consisting of a category C (the **underlying category**) and a subcategory $W \subset C$, the maps of which are called the **weak equivalences**, and which is *only* subject to the condition that it contains all the *objects* of C (and hence all the *identity maps*).

The category of small relative categories and the **relative** (i.e. weak equivalence preserving) functors between them will be denoted by \mathbf{RelCat} .

Two relative functors $C \rightarrow D$ are called **naturally weakly equivalent** if they can be connected by a finite zigzag of natural weak equivalences and a relative functor $f: C \rightarrow D$ will be called a **homotopy equivalence** if there exists a relative functor $g: D \rightarrow C$ (called a **homotopy inverse** of f) such that the compositions gf and fg are naturally weakly equivalent to 1_C and 1_D respectively.

3.2. What to look for in a generalization. In trying to generalize the notion of a relative category we were looking for

- * a notion of n -relative category for which the associated n -simplicial nerve functor to n -simplicial spaces, just like the classical nerve functor, has a *left adjoint* which is also a *left inverse*.

Motivated by the fact that in an n -simplicial space (i.e. an $(n+1)$ -simplicial set), just like in a simplicial space, the “space direction” plays a different role than “the n simplicial directions”, we start with considering sequences

$$\mathbf{C} = (a\mathbf{C}, v_1\mathbf{C}, \dots, v_n\mathbf{C}, w\mathbf{C}) \quad (n \geq 1)$$

consisting of a category $a\mathbf{C}$ and subcategories $v_1\mathbf{C}, \dots, v_n\mathbf{C}$ and $w\mathbf{C} \subset a\mathbf{C}$, each of which contains all the objects of $a\mathbf{C}$ and which together with $a\mathbf{C}$ form a commutative diagram with $2n$ arrows of the form

$$\begin{array}{ccc} & w\mathbf{C} & \\ \swarrow & & \searrow \\ v_1\mathbf{C} & \cdots & v_n\mathbf{C} \\ \searrow & & \swarrow \\ & a\mathbf{C} & \end{array}$$

Such a sequence can be considered to consist of n *relative categories* $v_1\mathbf{C}, \dots, v_n\mathbf{C}$ which each has the same *category of weak equivalences* $w\mathbf{C}$ and an *ambient category* $a\mathbf{C}$ which encodes the relations between the $v_i\mathbf{C}$ ($1 \leq i \leq n$).

However the associated n -simplicial nerve functor (4.2) will only recognize those *maps* in $a\mathbf{C}$ which are finite compositions of maps in the $v_i\mathbf{C}$ ($1 \leq i \leq n$) and only those *relations* which are a consequence of the commutativity of those squares in $a\mathbf{C}$ which are of the form

$$\begin{array}{ccc} \cdot & \xrightarrow{x_1} & \cdot \\ y_1 \downarrow & & \downarrow y_2 \\ \cdot & \xrightarrow{x_2} & \cdot \end{array}$$

in which $x_1, x_2 \in v_i\mathbf{C}$ and $y_1, y_2 \in v_j\mathbf{C}$ (where i and j are not necessarily distinct).

In order that the associated n -simplicial nerve functor has a left inverse we therefore have to impose some restrictions on $a\mathbf{C}$ and define as follows

3.3. n -Relative categories. An n -relative category \mathbf{C} will be an $(n+2)$ -tuple

$$\mathbf{C} = (a\mathbf{C}, v_1\mathbf{C}, \dots, v_n\mathbf{C}, w\mathbf{C})$$

consisting of a category $a\mathbf{C}$ and subcategories

$$v_1\mathbf{C}, \dots, v_n\mathbf{C} \text{ and } w\mathbf{C} \subset a\mathbf{C}$$

each of which contains all the objects of \mathbf{C} and which form a commutative diagram with $2n$ arrows of the form

$$\begin{array}{ccc} & w\mathbf{C} & \\ \swarrow & & \searrow \\ v_1\mathbf{C} & \cdots & v_n\mathbf{C} \\ \searrow & & \swarrow \\ & a\mathbf{C} & \end{array}$$

and where $a\mathbf{C}$ is subject to the condition that

- (i) every map in $a\mathbf{C}$ is a finite composition of maps in the $v_i\mathbf{C}$ ($1 \leq i \leq n$), and
- (ii) every relation in $a\mathbf{C}$ is a consequence of the commutativity of those squares in $a\mathbf{C}$ which are of the form

$$\begin{array}{ccc} \cdot & \xrightarrow{x_1} & \cdot \\ y_1 \downarrow & & \downarrow y_2 \\ \cdot & \xrightarrow{x_2} & \cdot \end{array}$$

in which $x_1, x_2 \in v_i\mathbf{C}$ and $y_1, y_2 \in v_j\mathbf{C}$ (where i and j are not necessarily distinct).

3.4. Some comments. In an n -relative category \mathbf{C} , the categories $v_1\mathbf{C}, \dots, v_n\mathbf{C}$ are *relative categories* which have $w\mathbf{C}$ as their category of weak equivalences, and we will therefore sometimes refer to the maps of $w\mathbf{C}$ as **weak equivalences**.

Moreover the category $a\mathbf{C}$ is more than a common underlying category for the $v_i\mathbf{C}$ ($1 \leq i \leq n$) (as it may contain additional relations) and will therefore be called the **ambient category**.

Also note that

- * A 1-relative category \mathbf{C} is essentially just an ordinary relative category, as in that case $a\mathbf{C} = v_1\mathbf{C}$.

3.5. Relative functors. A **relative functor** $f: \mathbf{C} \rightarrow \mathbf{D}$ between two n -relative categories \mathbf{C} and \mathbf{D} will be a functor $f: a\mathbf{C} \rightarrow a\mathbf{D}$ such that

$$fw\mathbf{C} \subset w\mathbf{D} \quad \text{and} \quad fv_i\mathbf{C} \subset v_i\mathbf{D} \quad \text{for all } i \leq n.$$

We will denote by $\mathbf{Rel}^n\mathbf{Cat}$ the resulting category of the small n -relative categories and the relative functors between them.

We end with

3.6. Some examples. Some rather simple but useful examples of n -relative categories are the following.

For every integer $p \geq 0$ let \mathbf{p} denote the category

$$0 \longrightarrow \cdots \longrightarrow p$$

and let $|\mathbf{p}| \subset \mathbf{p}$ be its subcategory which consists of the objects and their identity maps only. Then we will denote

(i) by $\mathbf{p}^w \in \mathbf{Rel}^n \mathbf{Cat}$ the object such that

$$a\mathbf{p}^w = v_i\mathbf{p}^w = w\mathbf{p}^w = \mathbf{p} \quad \text{for all } 1 \leq i \leq n$$

and

(ii) by $\mathbf{p}^{v_i} \in \mathbf{Rel}^n \mathbf{Cat}$ the object such that

$$a\mathbf{p}^{v_i} = v_i\mathbf{p}^{v_i} = \mathbf{p} \quad \text{and} \quad v_j\mathbf{p}^{v_i} = w\mathbf{p}^{v_i} = |\mathbf{p}| \quad \text{for } j \neq i.$$

A simple calculation then yields that, for every sequence of integers $p_n, \dots, p_1, q \geq 0$,

(iii) $\mathbf{p}_n^{v_n} \times \dots \times \mathbf{p}_1^{v_1} \in \mathbf{Rel}^n \mathbf{Cat}$ is such that

$$\begin{aligned} a(\mathbf{p}_n^{v_n} \times \dots \times \mathbf{p}_1^{v_1}) &= \mathbf{p}_n \times \dots \times \mathbf{p}_1 \\ w(\mathbf{p}_n^{v_n} \times \dots \times \mathbf{p}_1^{v_1}) &= |\mathbf{p}_n| \times \dots \times |\mathbf{p}_1| \quad \text{and} \\ v_i(\mathbf{p}_n^{v_n} \times \dots \times \mathbf{p}_1^{v_1}) &= |\mathbf{p}_n| \times \dots \times \mathbf{p}_i \times \dots \times |\mathbf{p}_1| \quad (1 \leq i \leq n) \end{aligned}$$

and

(iv) $\mathbf{p}_n^{v_n} \times \dots \times \mathbf{p}_1^{v_1} \times \mathbf{q}^w \in \mathbf{Rel}^n \mathbf{Cat}$ is such that

$$\begin{aligned} a(\mathbf{p}_n^{v_n} \times \dots \times \mathbf{p}_1^{v_1} \times \mathbf{q}^w) &= \mathbf{p}_n \times \dots \times \mathbf{p}_1 \times \mathbf{q} \\ w(\mathbf{p}_n^{v_n} \times \dots \times \mathbf{p}_1^{v_1} \times \mathbf{q}^w) &= |\mathbf{p}_n| \times \dots \times |\mathbf{p}_1| \times \mathbf{q} \quad \text{and} \\ v_i(\mathbf{p}_n^{v_n} \times \dots \times \mathbf{p}_1^{v_1} \times \mathbf{q}^w) &= |\mathbf{p}_n| \times \dots \times \mathbf{p}_i \times \dots \times |\mathbf{p}_1| \times \mathbf{q} \quad (1 \leq i \leq n) \end{aligned}$$

4. THE n -SIMPLICIAL NERVE FUNCTOR

We now

- introduce an adjunction (1.1 and 3.5)

$$K: s^n \mathbf{S} \longleftrightarrow \mathbf{Rel}^n \mathbf{Cat} : N$$

in which the right adjoint N is the n -simplicial nerve functor which we mentioned in 3.2,

- use N to lift the Reedy and the Rezk equivalences (1.1) from $s^n \mathbf{S}$ to $\mathbf{Rel}^n \mathbf{Cat}$,
- note that the unit of the above adjunction has two nice properties, and
- note that the counit is an isomorphism which is the same as saying that N has a *left adjoint* which is also a *left inverse* (cf. 3.2).

We start with

4.1. The adjunction $K: s^n \mathbf{S} \leftrightarrow \mathbf{Rel}^n \mathbf{Cat} : N$. The n -simplicial nerve functor will be the right adjoint in the adjunction (1.1 and 3.5)

$$K: s^n \mathbf{S} \longleftrightarrow \mathbf{Rel}^n \mathbf{Cat} : N$$

in which

(i) N sends an object $\mathbf{C} \in \mathbf{Rel}^n \mathbf{Cat}$ to the $(n+1)$ -simplicial set which as its (p_n, \dots, p_1, q) -simplices ($p_n, \dots, p_1, q \geq 0$) the maps (3.6)

$$\mathbf{p}_n^{v_n} \times \dots \times \mathbf{p}_1^{v_1} \times \mathbf{q}^w \longrightarrow \mathbf{C} \in \mathbf{Rel}^n \mathbf{Cat}$$

and

(ii) K is the colimit preserving functor which, for every $n+1$ integers $p_n, \dots, p_1, q \geq 0$, sends the *standard* (p_n, \dots, p_1, q) -simplex $\Delta[p_n, \dots, p_1, q]$ to

$$\mathbf{p}_n^{v_n} \times \dots \times \mathbf{p}_1^{v_1} \times \mathbf{q}^w \in \mathbf{Rel}^n \mathbf{Cat}.$$

Using the functor N we then can define

4.2. Reedy and Rezk equivalences in $\mathbf{Rel}^n \mathbf{Cat}$. A map $f \in \mathbf{Rel}^n \mathbf{Cat}$ will be called a **Reedy** or a **Rezk equivalence** if the map $Nf \in s^n \mathbf{S}$ is so (1.1), and we will denote by

$$\mathbf{Ry} \text{ and } \mathbf{Rk} \subset s^n \mathbf{S} \quad \text{and} \quad \mathbf{Ry} \text{ and } \mathbf{Rk} \subset \mathbf{Rel}^n \mathbf{Cat}$$

the subcategories of the Reedy and the Rezk equivalences in both $s^n \mathbf{S}$ and $\mathbf{Rel}^n \mathbf{Cat}$.

Next we note a very useful property of the functors K and N .

4.3. The 2-skeleton property.

- (i) *For every object $X \in s^n \mathbf{S}$, the n -relative category KX is completely determined by the 2-skeleton of X , i.e. the smallest subobject that contains all its multisimplices of total dimension ≤ 2 , and*
- (ii) *for every object $C \in \mathbf{Rel}^n \mathbf{Cat}$, the $(n+1)$ -simplicial set NC is completely determined by its 2-skeleton and is in fact its own 2-coskeleton.*

Proof. This follows by a straightforward calculation from the observation that the category (3.6)

$$a(\mathbf{p}_n^{v_n} \times \cdots \times \mathbf{p}_1^{v_1} \times \mathbf{q}^w) = \mathbf{p}_n \times \cdots \times \mathbf{p}_1 \times \mathbf{q}$$

is a poset which has an *object*, a *generating map* and a *relation* for every multisimplex of the standard (p_n, \dots, p_1, q) -simplex $\Delta[p_n, \dots, p_1, q]$ in total dimensions 0, 1 and 2 respectively.

The 2-skeleton property (4.3) readily implies

4.4. Proposition. *For every object $C \in \mathbf{Rel}^n \mathbf{Cat}$, the counit map*

$$\varepsilon C: KNC \longrightarrow C \in \mathbf{Rel}^n \mathbf{Cat}$$

is an isomorphism.

Moreover, in view of the fact that the composition

$$NC \xrightarrow{\eta NC} NKN C \xrightarrow{N\varepsilon C} NC \in s^n \mathbf{S}$$

is the identity, 4.4 implies

4.5. Proposition. *For every object $C \in \mathbf{Rel}^n \mathbf{Cat}$, the unit map*

$$\eta NC: NC \longrightarrow NKN C \in s^n \mathbf{S}$$

is an isomorphism.

We also note

4.6. Proposition. For every standard multisimplex $\Delta[p_n, \dots, p_1, q] \in s^n \mathbf{S}$, the unit map

$$\eta\Delta[p_n, \dots, p_1, q]: \Delta[p_n, \dots, p_1, q] \longrightarrow NK\Delta[p_n, \dots, p_1, q] \in s^n \mathbf{S}$$

is a Reedy equivalence.

Proof. Note that

$$\Delta[p_n, \dots, p_1, q] = \Delta[p_n, =] \times \cdots \times \Delta[=, p_i, =] \times \cdots \times \Delta[=, q]$$

where the $=$'s denote sequences of 0's and that

$$K\Delta[p_n, \dots, p_1, q] = \mathbf{p}_n^{v_n} \times \cdots \times \mathbf{p}_i^{v_i} \times \mathbf{q}^w.$$

A straightforward calculation then yields that

$$\eta\Delta[=, p_i, =]: \Delta[=, p_i, =] \longrightarrow NK\Delta[=, p_i, =] = N\mathbf{p}_i^{v_i} \in s^n \mathbf{S}$$

is an isomorphism for all $1 \leq i \leq n$, and that

$$\eta\Delta[=, q]: \Delta[=, q] \longrightarrow NK\Delta[=, q] \in s^n \mathbf{S}$$

is a Reedy equivalence, and the desired result now follows from the fact that N is a right adjoint and hence preserves products.

5. THE MAIN RESULT

Now we are ready for

- our main result, and
- a proof thereof, except for the verification of two propositions which we put off till §6.

We thus start with stating

5.1. Theorem.

- (i) *The relative functor* (4.1 and 4.2)

$$N: (\mathbf{Rel}^n \mathbf{Cat}, \mathbf{Rk}) \longrightarrow (s^n \mathbf{S}, \mathbf{Rk})$$

is a homotopy equivalence (3.1), *and hence*

- (ii) *the relative category* $(\mathbf{Rel}^n \mathbf{Cat}, \mathbf{Rk})$ *is, just like* $(s^n \mathbf{S}, \mathbf{Rk})$ (1.1), *a model for the homotopy theory of* n -fold homotopy theories.

To prove this, it suffices, in view of the fact that the Rezk equivalences in $s^n \mathbf{S}$ are the weak equivalences in a left Bousfield localization of the Reedy structure, to show

5.2. Theorem.

The relative functor (4.2)

$$N: (\mathbf{Rel}^n \mathbf{Cat}, \mathbf{Ry}) \longrightarrow (s^n \mathbf{S}, \mathbf{Ry})$$

is a homotopy equivalence (3.1).

In preparation for a proof we first discuss

5.3. The category of multisimplices. Let $\Delta[-] \subset s^n \mathbf{S}$ denote the full subcategory spanned by the standard multisimplices.

Given an object $X \in s^n \mathbf{S}$, one then defines its **category of multisimplices** ΔX as the over category

$$\Delta X = \Delta[-] \downarrow X .$$

Clearly this category is *natural in X*. Moreover it comes with a *forgetful functor*

$$F: \Delta X \longrightarrow s^n \mathbf{S}$$

which sends an object $\Delta[p_n, \dots, p_1, q] \rightarrow X$ to the object $\Delta[p_n, \dots, p_1, q] \in s^n \mathbf{S}$.

One then readily verifies that, as in the classical case, the resulting ΔX -diagram of standard multisimplices has the following properties.

5.4. Proposition. *For every object $X \in s^n \mathbf{S}$, the obvious map*

$$\operatorname{colim}_{\Delta X} F \longrightarrow X \in s^n \mathbf{S}$$

is an isomorphism.

5.5. Proposition. *For every object $X \in s^n \mathbf{S}$, the category ΔX is a Reedy category with fibrant constants [H, 15.10.1(2)].*

Next we introduce an n -relative version Δ_{rel} of the above functor Δ , which is the prospective homotopy inverse of the n -simplicial nerve functor.

5.6. The n -relative category of multisimplices functor. Let $\Delta_{\text{rel}}[-]$ denote the n -relative category such that

- (i) $a\Delta_{\text{rel}}[-] = \Delta[-]$ (5.3).
- (ii) $v_i\Delta_{\text{rel}}[-]$ ($1 \leq i \leq n$) is the subcategory of $\Delta[-]$ consisting of the maps

$$\Delta[p_n, \dots, p_1, q] \longrightarrow \Delta[p'_n, \dots, p'_1, q'] \in s^n \mathbf{S}$$

for which the associated map $\mathbf{p}_i \rightarrow \mathbf{p}'_i$ sends the object $p_i \in \mathbf{p}_i$ to the object $p'_i \in \mathbf{p}'_i$, and

- (iii) $w\Delta_{\text{rel}}[-] = v_1\Delta_{\text{rel}}[-] \cap \dots \cap v_n\Delta_{\text{rel}}[-]$.

Given an object $X \in s^n \mathbf{S}$ we then define its **n -relative category of multisimplices** $\Delta_{\text{rel}}X$ as the n -relative over category

$$\Delta_{\text{rel}}X = \Delta_{\text{rel}}[-] \downarrow X .$$

Clearly $\Delta_{\text{rel}}X$ is *natural in X* and the resulting functor

$$\Delta_{\text{rel}}: s^n \mathbf{S} \longrightarrow \mathbf{Rel}^n \mathbf{Cat}$$

has the following two properties which we will need in the proof of 5.2 but which will only be proved in 6.1–6.2 and 6.3–6.8 below respectively.

5.7. Proposition. *For every object $X \in s^n \mathbf{S}$, the obvious maps (5.3)*

$$\begin{aligned} \operatorname{colim}_{\Delta X} \Delta_{\text{rel}} F &\longrightarrow \Delta_{\text{rel}} X \in \mathbf{Rel}^n \mathbf{Cat}, & \text{and} \\ \operatorname{colim}_{\Delta X} N\Delta_{\text{rel}} F &\longrightarrow N\Delta_{\text{rel}} X \in s^n \mathbf{S} \end{aligned}$$

are isomorphisms.

5.8. Proposition. *There exists a natural transformation (4.1)*

$$\pi_t: \Delta_{\text{rel}} \longrightarrow K$$

with the property that, for every standard multisimplex $\Delta[p_n, \dots, p_1, q]$, the map

$$N\pi_t \Delta[p_n, \dots, p_1, q]: N\Delta_{\text{rel}} \Delta[p_n, \dots, p_1, q] \longrightarrow NK \Delta[p_n, \dots, p_1, q] \in s^n \mathcal{S}$$

is a Reedy equivalence.

Now we are ready for

5.9. A proof of theorem 5.2. To prove that the functors $N\Delta_{\text{rel}}$ and $1_{s^n \mathcal{S}}$ are naturally Reedy equivalent we consider, for every object $X \in s^n \mathcal{S}$, the commutative diagram (5.3, 5.6 and 5.8)

$$\begin{array}{ccccc} \text{colim}_{\Delta X} N\Delta_{\text{rel}} F & \xrightarrow{N\pi_t F} & \text{colim}_{\Delta X} NK F & \xleftarrow{\eta^F} & \text{colim}_{\Delta X} F \\ \downarrow & & \downarrow & & \downarrow \\ N\Delta_{\text{rel}} X & \xrightarrow{N\pi_t X} & NK X & \xleftarrow{\eta^X} & X \end{array}$$

in which the vertical maps are the obvious ones.

The vertical maps on the outside are, in view of 5.4 and 5.7, isomorphisms and it thus suffices to prove that the upper maps are Reedy equivalences. But this follows immediately from 5.8 and 4.6, and 5.5 and the result [H, 15.10.9(2)] that the colimit of an objectwise weak equivalence between Reedy cofibrant diagrams indexed by a Reedy category with fibrant constants is also a weak equivalence.

Note that the fact that these four maps are Reedy equivalences also implies that the functor $N\Delta_{\text{rel}}$ preserves Reedy equivalences and so does therefore (4.2) the functor Δ_{rel} .

To prove that the functors $\Delta_{\text{rel}} N$ and $1_{\mathbf{Rel}^n \mathbf{Cat}}$ are also naturally Reedy equivalent it suffices to show that, for every object $C \in \mathbf{Rel}^n \mathbf{Cat}$, both maps in the sequence

$$\Delta_{\text{rel}} NC \xrightarrow{\pi_t NC} KNC \xrightarrow{\varepsilon^C} C \in \mathbf{Rel}^n \mathbf{Cat}$$

are Reedy equivalences. For the second map this follows from 4.4. To deal with the first one we have to show that $N\pi_t NC$ is a Reedy equivalence in $s^n \mathcal{S}$. This we do by considering the above diagram for $X \in NC$

$$\begin{array}{ccccc} \text{colim}_{\Delta NC} N\Delta_{\text{rel}} F & \xrightarrow{N\pi_t F} & \text{colim}_{\Delta NC} NK F & \xleftarrow{\eta^F} & \text{colim}_{\Delta NC} F \\ \downarrow & & \downarrow & & \downarrow \\ N\Delta_{\text{rel}} NC & \xrightarrow{N\pi_t NC} & KNC & \xleftarrow{\eta^{NC}} & NC \end{array}$$

and then noting that all its maps are Reedy equivalences in view of the fact that

- (i) the upper and the outside vertical maps are so by the above,
- (ii) the map η^{NC} is so in view of 4.5, and
- (iii) Reedy equivalences have the two out of three property.

6. A PROOF OF PROPOSITIONS 5.7 AND 5.8

We start with

6.1. A proof of 5.7. Proposition 5.7 is a ready consequence of the more general proposition 6.2 below. To formulate the latter we will, for every pair of objects $C, D \in \mathbf{Rel}^n \mathbf{Cat}$, denote by $\text{map}(C, D)$ the set of maps $C \rightarrow D \in \mathbf{Rel}^n \mathbf{Cat}$. Then we can state

6.2. Proposition. *Let $T \in \mathbf{Rel}^n \mathbf{Cat}$ have an ambient category which is a poset with a terminal object T . Then, for every object $X \in s^n S$, the obvious map (5.3)*

$$\text{colim}_{\Delta X} \text{map}(T, \Delta_{\text{rel}} F) \longrightarrow \text{map}(T, \Delta_{\text{rel}} X)$$

is an isomorphism.

Proof. One readily verifies that the given map is onto. To show that it is also 1-1 we note that a map $T \rightarrow \Delta_{\text{rel}} X \in \mathbf{Rel}^n \mathbf{Cat}$ can be considered as a pair (f, x) of maps

$$T \xrightarrow{f} \Delta_{\text{rel}}[-] \in \mathbf{Rel}^n \mathbf{Cat} \quad \text{and} \quad fT \xrightarrow{x} X \in s^n S .$$

We then have to show that if

$$\begin{aligned} T &\xrightarrow{(f, z)} \Delta_{\text{rel}} \Delta[p_n, \dots, p_1, q] & \text{and} & \quad \Delta[p_n, \dots, p_1, q] \xrightarrow{y} X & \text{and} \\ T &\xrightarrow{(f', z')} \Delta_{\text{rel}} \Delta[p'_n, \dots, p'_1, q'] & \text{and} & \quad \Delta[p'_n, \dots, p'_1, q'] \xrightarrow{y'} X \end{aligned}$$

are such that

$$(f, yz) = (f', y'z') : T \longrightarrow \Delta_{\text{rel}} X ,$$

then these two pair represent the same element of $\text{colim}_{\Delta X} \text{map}(T, \Delta_{\text{rel}} F)$. This follows however from the observation that in that case $f = f'$ and that the following diagram commutes

$$\begin{array}{ccccc} \Delta[p_n, \dots, p_1, q] & \xleftarrow{z} & fT = f'T & \xrightarrow{z'} & \Delta[p'_n, \dots, p'_1, q'] \\ & \searrow y & \downarrow & \swarrow y' & \\ & & X & & \end{array}$$

6.3. A proof of 5.8. To prove proposition 5.8 takes more work.

The main part of the proof consists of proving (in 6.4–6.7 below) an identical statement for a functor

$$K_\delta : s^n S \longrightarrow \mathbf{Rel}^n \mathbf{Cat}$$

and then noting (in 6.8) that K_δ is essentially just an alternate way of describing the functor Δ_{rel} .

To do this we start with considering

6.4. The division of an n -relative category. Given an object $C \in \mathbf{Rel}^n \mathbf{Cat}$, its **division** $\delta C \in \mathbf{Rel}^n \mathbf{Cat}$ is defined as follows:

- (i) $a\delta C$ is the category which has as objects the functors $p \rightarrow aC$ ($p \geq 0$) and as maps

$$(x_1 : p_1 \rightarrow aC) \longrightarrow (x_2 : p_2 \rightarrow aC)$$

the commutative diagrams of the form

$$\begin{array}{ccc} p_1 & \xrightarrow{f} & p_2 \\ & \searrow x_1 \quad \swarrow x_2 & \\ & aC & \end{array}$$

and

- (ii) $v_i\delta C$ ($1 \leq i \leq n$) and $w\delta C$ consists of those maps as in (i) for which the induced map

$$x_1 p_1 = x_2 f p_1 \longrightarrow x_2 p_2$$

is in $v_i C$ or wC respectively.

Clearly δC is natural in C .

Moreover

- (iii) δC comes with a natural (**terminal**) **projection map**

$$\pi_t: \delta C \longrightarrow C \in \mathbf{Rel}^n \mathbf{Cat}$$

which sends each object $x: p \rightarrow C \in \delta C$ to the object $xp \in C$ and which clearly has the following property:

6.5. Proposition. *A map $f \in \delta C$ is in $v_i\delta C$ ($1 \leq i \leq n$) or $w\delta C$ iff $\pi_t f$ is in $v_i C$ or wC respectively.*

Using these divisions we then define

6.6. A functor $K_\delta: s^n S \rightarrow \mathbf{Rel}^n \mathbf{Cat}$ and a natural transformation $\pi_t: K_\delta \rightarrow K$. We denote by

$$K_\delta: s^n S \longrightarrow \mathbf{Rel}^n \mathbf{Cat}$$

the colimit preserving functor which sends each standard $\Delta[p_n, \dots, p_1, q]$ -simplex ($p_n, \dots, p_1, q \geq 0$) to the object

$$\delta p_n^{v_n} \times \dots \times \delta p_1^{v_1} \times \delta q^w \in \mathbf{Rel}^n \mathbf{Cat}$$

and with a slight abuse of notation we denote by

$$\pi_t: K_\delta \longrightarrow K$$

the natural transformation which is induced by the natural maps (6.4(iii))

$$(*) \quad \delta p_n^{v_n} \times \dots \times \delta p_1^{v_1} \times \delta q^w \xrightarrow{\pi_t \times \dots \times \pi_t \times \pi_t} p_n^{v_n} \times \dots \times p_1^{v_1} \times q^w$$

Now we can formulate the desired (6.3) variation on 5.8:

6.7. Proposition. *For every standard multisimplex $\Delta[p_n, \dots, p_1, q] \in s^n S$, the map (6.6)*

$$N\pi_t: NK_\delta \Delta[p_n, \dots, p_1, q] \longrightarrow NK \Delta[p_n, \dots, p_1, q] \in s^n S$$

is a Reedy equivalence.

Proof. As

$$K_\delta \Delta[p_n, \dots, p_1, q] = \delta \mathbf{p}_n^{v_n} \times \dots \times \delta \mathbf{p}_1^{v_1} \times \delta \mathbf{q}^w$$

and

$$K \Delta[p_n, \dots, p_1, q] = \mathbf{p}_n^{v_n} \times \dots \times \mathbf{p}_1^{v_1} \times \mathbf{q}^w$$

we have to prove that application of the functor N to the map in 6.6(*) above yields a Reedy equivalence. But N is a right adjoint and hence preserves products and it therefore suffices to show that each of the maps

$$N\pi_t: N\delta \mathbf{p}_i^{v_i} \longrightarrow N\mathbf{p}_i^{v_i} \quad \text{and} \quad N\pi_t: N\delta \mathbf{a}^w \longrightarrow N\mathbf{q}^w$$

is a Reedy equivalence.

To do this let

$$\tau: \mathbf{p}^{v_i} \longrightarrow \delta \mathbf{p}^{v_i} \quad \text{and} \quad \tau: \mathbf{p}^w \longrightarrow \delta \mathbf{p}^w$$

be the maps which send an object $b \in \mathbf{p}$ to the object

$$\mathbf{b} = (0 \rightarrow \dots \rightarrow b) \xrightarrow{\text{incl.}} (0 \rightarrow \dots \rightarrow p) \in \delta \mathbf{p}^{v_i} \text{ or } \delta \mathbf{p}^w.$$

Then $\pi_t \tau = 1$ and there are obvious maps

$$h: \delta \mathbf{p}^{v_i} \times \mathbf{1}^w \longrightarrow \delta \mathbf{p}_{v_i} \quad \text{and} \quad h: \delta \mathbf{p}^w \times \mathbf{1}^w \longrightarrow \delta \mathbf{p}^w$$

such that $h0 = 1$ and $h1 = \tau \pi_t$.

The desired result then follows readily from the observation that if

- (i) two maps $f, g: \mathbf{C} \rightarrow \mathbf{D} \in \mathbf{Rel}^n \mathbf{Cat}$ are *strictly homotopic* in the sense that there exists a map $h: \mathbf{C} \times \mathbf{1}^w \rightarrow \mathbf{D} \in \mathbf{Rel}^n \mathbf{Cat}$ connecting them,

then

- (ii) the maps $Nf, Ng: N\mathbf{C} \rightarrow N\mathbf{D} \in s^n \mathbf{S}$ are *strictly homotopic* in the sense that there exists a map $k: N\mathbf{C} \times \Delta[0, \dots, 0, 1] \rightarrow N\mathbf{D} \in s^n \mathbf{S}$ connecting them,

where

- (iii) k is the composition

$$\begin{aligned} N\mathbf{C} \times \Delta[0, \dots, 1] &\xrightarrow{\eta} N\mathbf{C} \times NK\Delta[0, \dots, 0, 1] \\ &\xrightarrow{\text{Id}} N\mathbf{C} \times N\mathbf{1}^w \xrightarrow{\approx} N(\mathbf{C} \times \mathbf{1}^w) \xrightarrow{h} N\mathbf{D} \end{aligned}$$

As mentioned in 6.3 above, proposition 5.8 now is an immediate consequence of 6.7 above and

6.8. Proposition. *There exists a commutative diagram*

$$\begin{array}{ccc} \Delta_{\text{rel}} & \xrightarrow{\quad} & K_\delta \\ & \searrow & \swarrow \pi_t \\ & K & \end{array}$$

of functors $s^n \mathbf{S} \rightarrow \mathbf{Rel}^n \mathbf{Cat}$ and natural transformations between them in which

- (i) the right hand map is as in 6.6 and

- (ii) *the top map is an isomorphism which, for every standard multisimplex $\Delta[p_n, \dots, p_1, q] \in s^n \mathbf{S}$ sends*

$$\Delta_{\text{rel}} \Delta[p_n, \dots, p_1, q] \quad \text{to} \quad K_\delta \Delta[p_n, \dots, p_1, q] .$$

Proof. As both functors Δ_{rel} and K_δ are colimit preserving (5.7 and 6.6) this follows immediately from the observation that, for every sequence of integers $p_n, \dots, p_1, q \geq 0$

$$\Delta_{\text{rel}} \Delta[p_n, \dots, p_1, q] \quad \text{and} \quad \delta p_n^{v_n} \times \dots \times \delta p_1^{v_1} \times \delta q^w$$

are canonically isomorphic.

7. APPENDIX

In this appendix we note that the categories $\mathbf{Rel}^n \mathbf{Cat}$ ($n \geq 1$) have two additional properties which one would expect a homotopy theory of homotopy theories to have:

- A. There there exists a functor $\mathbf{Rel}^n \mathbf{Cat} \rightarrow \mathbf{Rel}^{n+1} \mathbf{Cat}$ which has a *left inverse right adjoint*.
- B. That every object of $\mathbf{Rel}^{n+1} \mathbf{Cat}$ gives rise to a *category enriched over $\mathbf{Rel}^n \mathbf{Cat}$* which suggests the possibility that “a map in $\mathbf{Rel}^{n+1} \mathbf{Cat}$ is a Rezk equivalence (4.2) iff the induced map between these enriched categories is a kind of *DK-equivalence*”.

To deal with A we note that a straightforward calculation yields:

7.1. Proposition. *For every integer $n \geq 1$ the functor*

$$\mathbf{Rel}^n \mathbf{Cat} \longrightarrow \mathbf{Rel}^{n+1} \mathbf{Cat}$$

which sends

$$(aC, v_1 C, \dots, v_n C, wC) \quad \text{to} \quad (aC, v_1 C, \dots, v_n C, wC, wC)$$

has a right adjoint left inverse which sends

$$(aD, v_1 D, \dots, v_{n+1} D, wD) \quad \text{to} \quad (\bar{a}D, v_1 D, \dots, v_n D, wD)$$

where $\bar{a}D \subset aD$ denotes the subcategory which consists of the finite compositions of maps in the $v_i D$ ($1 \leq i \leq n$).

We deal with B by means of an n -relative version of the Grothendieck enrichment of [DHKS, 3.4 and 3.5].

To do this we start with recalling

7.2. Types of zigzags. The **type** of a zigzag of maps in a category C from an object X to an object Y

$$X \xrightarrow{f_1} \dots \xrightarrow{f_m} Y \quad (m \geq 0)$$

will be the pair $T = (T_+, T_-)$ of complementary subsets of the set of integers $\{1, \dots, m\}$ such that $i \in T_+$ whenever f_i is a forward map and $i \in T_-$ otherwise.

These types can be considered as the objects of a **category of types T** which has, for every two types (T_+, T_-) and (T'_+, T'_-) of length m and m' respectively, as maps $t: (T_+, T_-) \rightarrow (T'_+, T'_-)$ the weakly monotonic maps $t: \{1, \dots, m\} \rightarrow \{1, \dots, m'\}$ such that

$$tT_+ \subset T'_+ \quad \text{and} \quad tT_- \subset T'_- .$$

With these types one then associates

7.3. n -Relative arrow categories. Given an object $\mathbf{C} \in \mathbf{Rel}^{n+1}\mathbf{Cat}$ let, as in 7.1, $\bar{a}\mathbf{C} \subset a\mathbf{C}$ denote the subcategory which consists of the finite compositions of maps of the $v_i\mathbf{C}$ ($1 \leq i \leq n$).

For every pair of objects $X, Y \in \mathbf{C}$ and type T (7.2) we then denote by $\mathbf{C}^T(X, Y) \in \mathbf{Rel}^n\mathbf{Cat}$ the n -relative arrow category which has

- (i) as objects the zigzags of type T in \mathbf{C} between X and Y in which the backward maps are in $\bar{a}\mathbf{C}$,
- (ii) as maps in $v_i\mathbf{C}^T(X, Y)$ ($1 \leq i \leq n$) and $w\mathbf{C}^T(X, Y)$ between two such zigzags the commutative diagrams of the form

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & \cdot & \cdots & \cdot & \xrightarrow{\quad} & Y \\ \downarrow 1 & & \downarrow & & \downarrow & & \downarrow 1 \\ X & \xrightarrow{\quad} & \cdot & \cdots & \cdot & \xrightarrow{\quad} & Y \end{array}$$

in which the vertical maps are in $v_i\mathbf{C}$ and $w\mathbf{C}$ respectively, and

- (iii) as maps in $a\mathbf{C}^T(X, Y)$ the finite compositions of maps of the $v_i\mathbf{C}^T(X, Y)$ ($1 \leq i \leq n$).

These arrow categories in turn give rise to

7.4. T -diagrams of arrow categories. Given an object $\mathbf{C} \in \mathbf{Rel}^n\mathbf{Cat}$ and objects $X, Y \in \mathbf{C}$, one can form a **T -diagram of arrow categories**

$$\mathbf{C}^{(T)}(X, Y): \mathbf{T} \longrightarrow \mathbf{Rel}^n\mathbf{Cat}$$

which assigns to every object $T \in \mathbf{T}$ the arrow category

$$\mathbf{C}^T(X, Y) \in \mathbf{Rel}^n\mathbf{Cat}$$

and to every map $t: T \rightarrow T' \in \mathbf{T}$ the map

$$t_*: \mathbf{C}^T(X, Y) \longrightarrow \mathbf{C}^{T'}(X, Y) \in \mathbf{Rel}^n\mathbf{Cat}$$

which sends a zigzag of type T

$$X \xrightarrow{f_1} \cdot \cdots \cdot \xrightarrow{f_m} Y$$

to the zigzag of type T'

$$X \xrightarrow{f'_1} \cdot \cdots \cdot \xrightarrow{f'_m} Y$$

in which each f'_j ($1 \leq j \leq m'$) is the composition of the f_i with $ti = j$ or, in no such i exists, the appropriate identity map.

Now we can form

7.5. The Grothendieck construction on $\mathbf{C}^{(T)}(X, Y)$. Given an object $\mathbf{C} \in \mathbf{Rel}^{n+1}\mathbf{Cat}$ and objects $X, Y \in \mathbf{C}$ the **Grothendieck construction** on $\mathbf{C}^{(T)}(X, Y)$ is the object

$$\mathrm{Gr} \mathbf{C}^{(T)}(X, Y) \in \mathbf{Rel}^n \mathbf{Cat}$$

which has

- (i) as objects the zigzags in \mathbf{C} between X and Y in which the backward maps are in $\bar{a}\mathbf{C}$, i.e. pairs (T, Z) consisting of objects

$$T \in \mathbf{T} \quad \text{and} \quad Z \in \mathbf{C}^T(X, Y)$$

and

- (ii) for every two such objects (T, Z) and (T', Z') , as maps $(T, Z) \rightarrow (T', Z')$ the pairs (t, z) consisting of maps

$$t: T \longrightarrow T' \in \mathbf{T} \quad \text{and} \quad z: t_* Z \longrightarrow Z' \in \mathbf{C}^{T'}(X, Y)$$

and in which

- (iii) for every two composable maps (t, z) and (t', z') their composition is defined by the formula

$$(t', z')(t, z) = (t't, z'(t_* z))$$

Together these Grothendieck constructions give rise to

7.6. A Grothendieck enrichment. Given an object $\mathbf{C} \in \mathbf{Rel}^{n+1}\mathbf{Cat}$ we now define its **Grothendieck enrichment** as the category $\mathrm{Gr} \mathbf{C}^{(T)}$ enriched over $\mathbf{Rel}^n \mathbf{Cat}$ which

- (i) has the same objects as \mathbf{C} ,
- (ii) has for every two objects $X, Y \in \mathbf{C}$, as its hom-object the n -relative category $\mathbf{C}^{(T)}(X, Y)$, and
- (iii) has, for every three objects X, Y and $Z \in \mathbf{C}$ as composition

$$\mathrm{Gr} \mathbf{C}^{(T)}(X, Y) \times \mathrm{Gr} \mathbf{C}^{(T)}(X, Y) \longrightarrow \mathrm{Gr} \mathbf{C}^{(T)}(X, Z)$$

the function induced by the compositions of the zigzags involved.

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